Strategy For Testing Whether a Given Vector Field is Conservative

Given $\vec{F}(x,y) = \langle h(x,y), g(x,y) \rangle$, compute $\frac{\partial h}{\partial y}$ and $\frac{\partial g}{\partial x}$. If these are not equal, $\vec{F}$ is not conservative. If they are equal, try to integrate to find f such that $\vec{F} = \nabla f$

Example 1: Are the following vector fields conservative?

a) $\langle 17\sin(x), 10x-3y \rangle$

b)

$\langle 2xy\sec^2(yx^2)+3x^2, x^2\sec^2(yx^2)+y \rangle$

a) $h(x,y) = 17\sin(x)$

$g(x,y) = 10x - 3y$

$\frac{\partial h}{\partial y} = 0$, $\frac{\partial g}{\partial x} = 10$

$0 \neq 10$, so <u>not</u>

conservative

b) $h(x,y) = 2xy\sec^2(yx^2) + 3x^2$

$g(x,y) = x^2\sec^2(yx^2) + y$

$$\frac{\partial n}{\partial y} = 2x\sec^2(yx^2) + 4x^3y\sec^2(yx^2)\tan(yx^2)$$

$$\frac{\partial g}{\partial x} = 2x\sec^2(yx^2) + 4x^3y\sec^2(yx^2)\tan(yx^2)$$

these are equal, so may

be conservative

Integrate

$h(x,y)$ with respect to x

get $\boxed{f(x,y) = \tan(yx^2) + x^3 + k(y)}$

for some function k.

Similarly, integrating $g(x,y)$

with respect to y yields

$$\boxed{f(x,y) = \tan(yx^2) + \frac{y^2}{2} + m(x)}$$

for some function m

Equating these two,

$$f(x,y) = \tan(yx^2) + \frac{y^2}{2} + m(x)$$
$$= \tan(yx^2) + k(y) + x^3 .$$

Then $\frac{y^2}{2}$ could equal $k(y)$ and

x^3 could equal $m(x)$, so

we can take

$$\boxed{f(x,y) = \tan(yx^2) + \frac{y^2}{2} + x^3}$$

and so the vector field is conservative

Example 2. Compute

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{where}$$

$$\vec{F}(x,y) = \Big\langle \underbrace{\frac{y}{x}}_{P}, \underbrace{\ln(x)}_{Q} \Big\rangle$$

and C is the curve parameterized by

$\vec{r}(t) = \langle e^t, e^{-t} \rangle$ from

$t = 0$ to $t = 1$

Note $\frac{\partial P}{\partial y} = \frac{1}{x}$

$$\frac{\partial Q}{\partial x} = \frac{1}{x}$$

so $\vec{F}$ could be conservative.

After integrating as in the previous example, we'd get $\vec{F} = \nabla f$

where $f(x,y) = y\ln(x)$,

for example

By the Fundamental Theorem,

$$\int_C \vec{F}(x,y)\cdot d\vec{r}$$

$$= \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

(remember r starts at $t=0$ and ends at $t=1$)

$$= f\left(e, \frac{1}{e}\right) - f(1,1)$$

$$= \frac{1}{e}\ln(e) - \ln(1)$$

$$= \boxed{\frac{1}{e}}$$

Example 3: (the silo problem)

#46, book

Equation of path:

$$\langle 20\cos(6\pi t), 20\sin(6\pi t), 90t \rangle$$

$$0 \le t \le 1$$

Force = force of gravity, which is completely vertical

Since the man weighs 160 pounds and paint leaks out at a steady rate of 9 pounds, an equation for the force would be

$$\vec{F} = \langle 0, 0, 185 - \frac{z}{10} \rangle$$

Could do this using Fundamental Theorem, but...

$r'(t) = \langle -120\pi \sin(6\pi t), 120\pi \cos(6\pi t), 90 \rangle$,

$\vec{F}(r(t)) = \langle 0, 0, 185 - 9t \rangle$

$\vec{F}(r(t)) \cdot r'(t)$

$= 90(185 - 9t)$.

So the work is

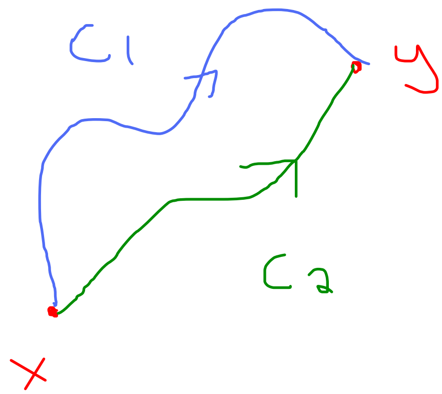
$$\int_0^1 90(185-9t)\,dt$$

$$= 90\left(185t - \frac{9t^2}{2}\right)\Big|_0^1 = \boxed{16245 \text{ ft lbs}}$$

Fixing two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$, if $\vec{F}$ is a vector field on $\mathbb{R}^n$ with all component functions continuous and C_1, C_2 are two "nice enough" curves in $\mathbb{R}^n$ that both start at x and end at y, we can ask whether

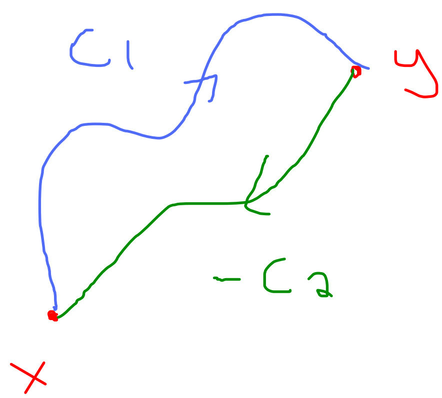
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

But note

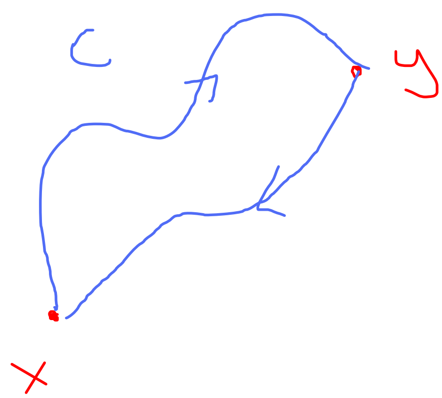


forms a "closed curve" C (same start and finish points) by first going through C_1, then $-C_2$ (opposite orientation)

We have



and then



Since

$$\int_{-C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r},$$

we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

is the same as

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0,$$

which is the same as $\int_{C} \vec{F} \cdot d\vec{r} = 0$

We've then got the following result:

$\int_{C_1} \vec{F} \cdot d\vec{r}$ being independent of the curve C_1 is the same as $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C

Example 4: Show that

$$\vec{F}(x,y) = \langle \underbrace{3x-4y}_{P}, \underbrace{x-2y}_{Q} \rangle$$

is not path-independent for paths from $(0,0)$ to $(1,2)$

Since $\dfrac{\partial P}{\partial y} = -4$, $\dfrac{\partial Q}{\partial x} = 1$,

we see $\vec{F}$ is not conservative,

so we can't use the Fundamental Theorem

Just brute-force calculate.

let C_1 be $y=2x$ and

C_2 be $y=x^2+x$.

Both curves have $(0,0)$ and

$(1,2)$ on their graph.

Parameterize C_1 by

$\vec{r}_1(t)=\langle t, 2t\rangle$, $0 \le t < 1$, and C_2

by $\vec{r}_2(t)=\langle t, t^2+t\rangle$,

$0 \le t \le 1$.

Then

$$\int_{C_1} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle 3t - 8t, t - 4t \rangle \cdot \overbrace{\langle 1, 2 \rangle}^{r'(t)} \, dt$$

$$= \int_0^1 (-5t - 6t) \, dt$$

$$= -11 \int_0^1 t \, dt = \boxed{-\frac{11}{2}}$$

But

$$\int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \langle 3t-4(t^2+t),\ t-2(t^2+t)\rangle \cdot \underbrace{\langle 1, 2t+1\rangle}_{r_2'(t)}\, dt$$

$$= \int_0^1 \langle -t-4t^2,\ -t-2t^2\rangle \cdot \langle 1, 2t+1\rangle\, dt$$

$$= \int_0^1 (-t-4t^2-2t^2-4t^3-2t^2-t)\, dt$$

$$= \int_0^1 (-4t^3-8t^2-2t)\, dt$$

$$= \left(-t^4-\frac{8t^3}{3}-t^2\right)\Big|_0^1 = \boxed{-\frac{14}{3}}$$

Since $-14/3 \neq -11/2$,

$\vec{F}$ is not independent of path.

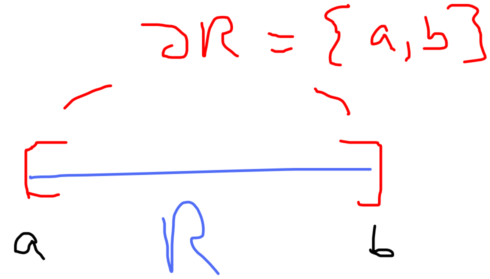
Green's Theorem (16.4)

Fundamental Theorem of Calculus – adolescent version.

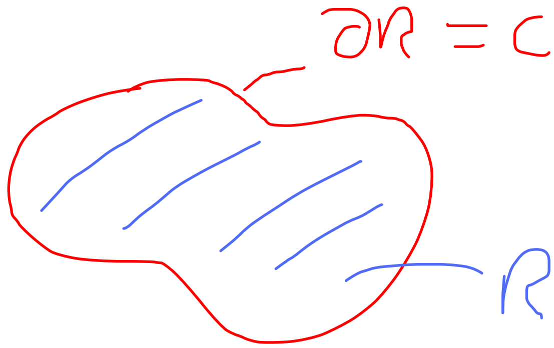
Given a region R in $\mathbb{R}^2$ with boundary curve C, we want to relate an integral over R to an integral over C

Picture

1-D



2-D



3-D Later!

Green's Theorem:

Let R be a region in $\mathbb{R}^2$ with boundary curve C parameterized by

$r(t) = \langle x(t), y(t) \rangle$

for $a \leq t \leq b$ in the counterclockwise orientation

Suppose

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

is a vector field satisfying

1) P, Q continuous on R

2) $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial Q}{\partial y}$ continuous

in an open region containing R

Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_C \vec{F}(x,y) \cdot d\vec{r}$$

$$= \int_C \left(P(x,y)\,dx + Q(x,y)\,dy \right)$$

Green's Theorem